数理解析研究所講究録
574巻 1985年 51-57

A Very High-dimensional Chaotic Attractor

池 田 研 介
Kensuke Ikeda

Research Institute for Fundamental Physics
Kyoto University, Kyoto 606, Japan

and
松本 健司
Kenji Matsumoto

Faculty of Pharmaceutical Sciences
Hokkaido University, Sapporo 060, Japan

ABSTRACT

The nature of a very high-dimensional chaotic attractor is investigated for the purpose of elucidating the relationships between the physical processes of turbulent behavior occuring in the real space and global characteristics of attractor in the phase space.

1. INTRODUCTION

The aim of the present study is to elucidate how very complicated turbulent behaviors observed in physical space is related with structure of attractors in the phase space by analyzing a simple infinite dimensional dissipative system as an example. In particular, we show that the qualitative difference between in the "interior" and in the "exterior" of the chaotic attractor is reflected in the properties of appropriately processed time series (or spatial pattern).

* Details of the present article are to be submitted to J. Stat. Phys.

The system we are to analyze is a model of optical turbulence described by the delay-differential equation such that

$$\dot{x}(t) = -x(t) + \pi\mu f(x(t - t_R)) , \tag{1}$$

with

$$f(x) = \sin(x - x_0) \tag{2}$$

Eq.(1) is a simplified version of a space-time nonlinear partial=differential equation under an appropriate boundary condition[1), and it can be interpreted as a mapping rule from a "spatial pattern" $x_n(\tau) \equiv x(\tau + nt_R)$ $(0 < \tau \leqq t_R)$ to a new pattern $x_{n+1}(\tau)$ after a finite time interval t_R:

$$x_{n+1}(\tau) = x_n(t_R)e^{-\tau} + \pi\mu\int_0^{\tau} ds\, e^{-(\tau - s)} f(x_n(s))ds . \tag{3}$$

Consider an infinite dimensional vector $\vec{R}$ whose component at τ $(0 < \tau \leqq t_R)$ is given by $x_n(\tau)$. Then eq.(3) defines a mapping rule $\vec{R}_n \xrightarrow{F} \vec{R}_{n+1}$. We call $\vec{R}_n$ the state vector.

To observe the chaotic behavior in the phase space we introduce two kinds of fundamental bases; one is the Lyapunov basis composed of the Lyapunov vectors $\{\vec{e}_i(\vec{R}')\}$ defined at an arbitrary point $\vec{R}'$ in the attractor $\hat{A}$, and another is the Fourier basis composed of the Fourier modes $\vec{f}_k$, the component of which is expressed as $\sqrt{2/t_R}\cos\omega_k\tau$ (k: even) or $\sqrt{2/t_R}\sin\omega_{k+1}\tau$ (k: odd) at $\tau(0 < \tau \leqq t_R)$ where $\omega_k \equiv k\pi/t_R$. The Lyapunov vector is numbered in order of the magnitude of the corresponding Lyapunov exponent λ_i i.e., $\lambda_1 > \lambda_2 > \cdots$.

2. LYAPUNOV EXPONENTS AND LYAPUNOV COMPONENTS

Consider the linear fluctuation around the stationary solution of eq.(1). The linear fluctuation mode is very similar to the Fourier mode introduced above. The smaller frequency modes are unstable, contributing the chaotic behavior, however, the higher frequency modes are stabilized and specified by negative decay rate asymptotically:

$$\alpha_i \underset{i \to \infty}{\to} -\log i + \text{const} \quad , \tag{4}$$

where i indicates the mode number. An interesting fact is that the Lyapunov exponent is closely related with α_i: as the order i exceeds some characteristic order I_{LE}, the i-th Lyapunov exponent λ_i coincides with α_i. Moreover, I_{LE} is closely related with the Lyapunov dimension D of Kaplan and York

$$I_{LE} = a_E D \quad , \tag{5}$$

where a_E is estimated to be 0.65 or so.

The averaged Lyapunov component defined by

$$C_i = [\langle\langle(\vec{R}.\ \vec{e}_i(\vec{R}'))^2\rangle\rangle_{\vec{R},\ \vec{R}' \in \hat{A}}]^{1/2} \tag{6}$$

characterizes the global size of attractor measured along the i-th Lyapunov vector. Results of numerical simulation reveal that C_i exhibits quite similar behavior as the quantity e^{λ_i} characterizing the local width of attractor:

$$C_i = \exp \eta\lambda_i \quad , \tag{7}$$

where the exponent η weakly dependent upon i for i < D takes a constant value for i > D. Therefore, there is a characteristic order $i = I_{LC}$ corresponding to I_{LE} above which

$$C_i \to \exp \eta\alpha_i = i^{-\eta} \quad . \tag{8}$$

Thus the Lyapunov component decays algebraically. I_{LC} is slightly larger than I_{LE}; empirically we have

$$I_{LC} = a_c D + b_c \quad , \tag{9}$$

where $a_c \approx .70$ and $b_c \approx 6.0$.

3. PROJECTION ONTO EXTERIORS

In our system the order of Lyapunov vectors $\{\vec{e}_i(\vec{R}')\}$ are almost conserved as $\vec{R}'$ is moved over the attractor $\hat{A}$. This implies that the subspace $EX_\ell^L(\vec{R}')$ spanned by the vectors $\vec{e}_\ell(\vec{R}')$, $\vec{e}_{\ell+1}(\vec{R}')$, $\cdots$ excludes the attractor if the order ℓ is chosen to be larger than D. Such subspaces $EX_\ell^L(\vec{R}')$ form the "exterior" of attractor. Thus we call $EX_\ell^L(\vec{R}')$ the exterior of the ℓ-th order. Our concern is how the projection of the chaotic motion onto the exterior $EX_\ell^L(\vec{R}')$ changes as ℓ is increased from far below to far above D: Let $\psi_\ell(\tau,\vec{R}|\vec{R}')$ be the component at τ of the projection of the state vector $\vec{R}$ onto $EX_\ell^L(\vec{R}')$. Then the correlation between the two time series $\psi_\ell^2(\tau,\vec{R}|\vec{R}')$ and $\psi_m^2(\tau,\vec{R}|\vec{R}')$ changes drastically in such a way that the number of ψ_m^2 correlated with ψ_ℓ^2 $(m \neq \ell)$ increases drastically as ℓ exceeds the characteristic order I_{LC} introduced in 1. and eventually the exterior EX_ℓ^L goes out of the attractor. The principal origin of such a behavior is the transition to the algebraic decay of eq.(8) which occurs when the order exceeds I_{LC}. An another origin is the localization of the time series $\psi_\ell^2(\tau,\vec{R}|\vec{R}')$ which is enhanced as ℓ increases.

4. LYAPUNOV-FOURIER CORRESPONDENCE

A remarkable feature of our attractor is that each of the

Lyapunov vector have one to one correspondence with each of the Fourier vector in the following sense: The Fourier vector $\vec{f}_k$ is constructed by the Lyapunov vectors $\vec{e}_i(\vec{R}')$ whose order i is in the vicinity of k for almost all $\vec{R}'$ over the attractor (and vice versa). Thus we can expect that a similar phenomenon as has been observed for the projection onto the exterior subspace $EX^L_\ell(\vec{R}')$ defined for the Lyapunov basis will be seen also on the Fourier basis. Let us define the subspace EX^F_k spanned by the Fourier vectors $\vec{f}_k, \vec{f}_{k+1}, \vec{f}_{k+2}, \cdots$, which we denote by EX^F_k. Then the projection of $\vec{R}$ onto $EX^L_\ell(\vec{R}')$ corresponds to the projection of $\vec{R}$ onto EX^F_k. We may conjecture that the order k corresponds to the order ℓ in the following manner: Consider $P_{LF}(i|k) \equiv C_i^2 \langle\langle(\vec{e}_i(\vec{R}').\vec{f}_k)^2\rangle\rangle_{\vec{R}' \in \hat{A}}$. Let us denote i that maximizes $P_{LF}(i|k)$ for a given k by $i^*(k)$. Then

$$k = i^{*-1}(\ell) \quad . \tag{10}$$

The projection of $\vec{R}$ onto EX^F_k has a clear physical meaning. It is nothing but the high-pass filtered time series of the lower cutoff frequency $\omega_k = (\pi/t_R)k$.

5. HIGH-PASS FILTERED TIME SERIES

It is known that the high-pass filtered time series

$$\phi_k(\vec{R},\tau) = \sum_{q \geq k} F_q \exp i\omega_q\tau \quad , \tag{11}$$

where $F_R \equiv \vec{f}_k.\vec{R}$, in general becomes localized into several "active" time domains as the cutoff frequency ω_k increases (intermittency). In the active domain the time series $\phi_k^2(\vec{R},\tau)$ looks like bursts of random fluctuation. In Fig.1 we show how the active time domains indicated by lines are distributed over the two dimensional plane of τ and ω. The distribution of active domains has quite a different pattern in the high frequency regime from the one in the low frequency regime; in the former regime active domains at different frequencies are connected with each other, whereas in the latter regime such connectivity disappears and the distribution of active domains forms an irregular pattern. The above property can be quantitatively described by the correlation between the two high-pass filtered time series. Let $\Delta_F(\omega)$ be the width of frequency range in which $\phi^2_{k'}(\vec{R},\tau)$ is significantly correlated with $\phi^2_k(\vec{R},\tau)$ $(k = \omega t_R/\pi)$. Fig.2 shows an example of $\Delta_F(\omega)$, which divergently increases for ω beyond a characteristic ω_c. The frequency ω_c is the boundary of the two different patterns mentioned above, and it should be the counterpart of I_{LC} in the Fourier basis. Indeed we have ascertained that $K_{FC} \equiv \omega_c t_R/\pi$ is related with I_{LC} through the correspondence rule of eq.(9) for various sets of parameters (t_R,μ).

In conclusion we have shown that various quantities characterizing the attractor exhibit notable changes as we go from the interior into the exterior of attractor. Such behaviors are clearly reflected in the statistical properties of the high-pass filtered time series through the Lyapunov-Fourier correspondence. It is strongly desired to examine whether these observations are valid also for other classes of high-dimensional attractors.

REFERENCE

1) K. Ikeda, Opt. Commun. 30, 257 (1979) ; K. Ikeda, H. Daido and O. Akimoto, Phys. Rev. Lett. 45, 709 (1980) ; K. Ikeda and M. Mizuno, to be published in the special issue of IEEE J. of Quantum Electron on optical bistability

FIG.1

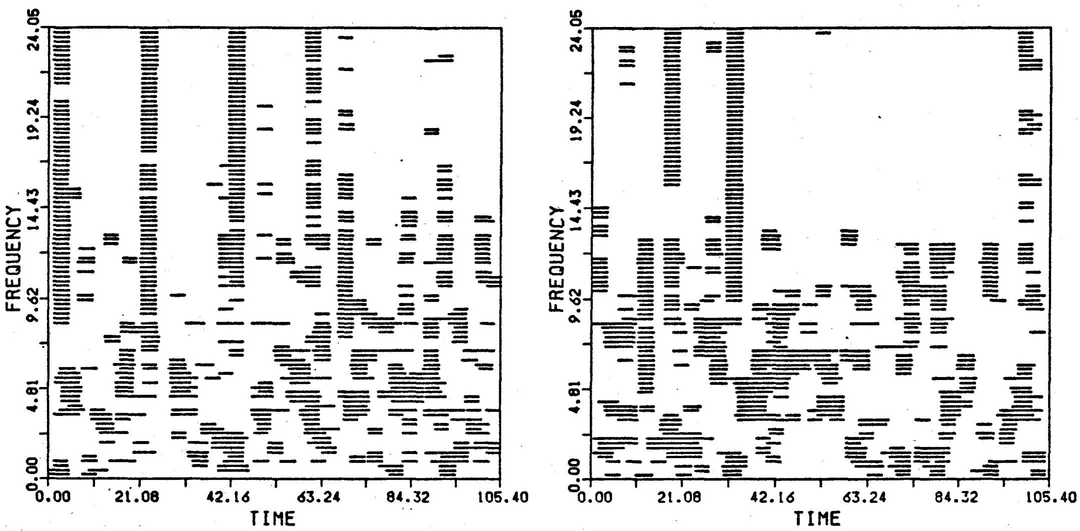

FIG.2